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EPR entangled states and complex fractional Fourier transformation*

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Abstract. Starting from the Einstein-Podolsky-Rosen entangled state representations of continuous variables we derive a new formulation of complex fractional Fourier transformation (CFFT). We find that two-variable Hermite polynomials are just the eigenmodes of the CFFT. In this way the CFFT is linked to the appropriate operator transformation between two kinds of entangled states in the context of quantum mechanics. In so doing, the CFFT of quantum mechanical wave functions can be derived more directly and concisely.

PACS. 03.65.Ud Entanglement and quantum nonlocality (e.g. EPR paradox, Bell's inequalities, GHZ states, etc.) – 42.30.Lr Modulation and optical transfer functions – 42.50.Dv Nonclassical field states; squeezed, antibunched, and sub-Poissonian states; operational definitions of the phase of the field; phase measurements

1 Introduction

Since the publication of the paper of Einstein, Podolsky and Rosen (EPR) in 1935 [1], arguing the incompleteness of quantum mechanics, the conception of entanglement has become more and more fascinating and important, though it is also weird. EPR noticed that in an entangled state, measurement performed on one part of a bipartite system provides information on the remaining part. In reference [2] according to the original idea of EPR that two particles' relative coordinate operator commutes with their total momentum operator, $[X_1 - X_2, P_1 + P_2] = 0$, Fan and Klauder have constructed the entangled state representation in two-mode Fock space, which is the common eigenvector of $X_1 - X_2$ and $P_1 + P_2$, *i.e.*,

$$|\eta\rangle = \exp\left[-\frac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right]|00\rangle,$$
$$\eta = \eta_1 + \mathrm{i}\eta_2 = |\eta|\mathrm{e}^{\mathrm{i}\varphi}, \quad (1)$$

where $|00\rangle$ is the vacuum state, Bose operators $(a_1^\dagger, a_2^\dagger)$ are related to X_i and P_i, $(i = 1, 2)$ by $X_i = (a_i + a_i^\dagger)/\sqrt{2}$, $P_i = (a_i - a_i^\dagger)/\sqrt{2}\,\mathrm{i}$. Using $[a_i, a_j^\dagger] = \delta_{ij}$, we can show

$$\left(a_1 - a_2^\dagger\right)|\eta\rangle = \eta\,|\eta\rangle, \quad \left(a_2 - a_1^\dagger\right)|\eta\rangle = -\eta^*\,|\eta\rangle. \quad (2)$$

$\eta' s$ real and imaginary part are the eigenvalue of $X_1 - X_2$ and $P_1 + P_2$, respectively, *i.e.*,

$$(X_1 - X_2)\,|\eta\rangle = \sqrt{2}\eta_1\,|\eta\rangle, \qquad (P_1 + P_2)\,|\eta\rangle = \sqrt{2}\eta_2\,|\eta\rangle. \quad (3)$$

On the other hand, we have introduced another EPR entangled state $|\xi\rangle$,

$$|\xi\rangle = \exp\left[-\frac{|\xi|^2}{2} + \xi a_1^\dagger + \xi^* a_2^\dagger - a_1^\dagger a_2^\dagger\right]|00\rangle,\ \xi = \xi_1 + \mathrm{i}\xi_2, \quad (4)$$

which is the common eigenvector of $P_1 - P_2$ and $X_1 + X_2$,

$$(X_1 + X_2)\,|\xi\rangle = \sqrt{2}\,\xi_1\,|\xi\rangle, \quad (P_1 - P_2)\,|\xi\rangle = \sqrt{2}\,\xi_2\,|\xi\rangle. \quad (5)$$

Due to

$$[X_1 - X_2,\ P_1 - P_2] = 2\mathrm{i},\ \ [X_1 + X_2, P_1 + P_2] = 2\mathrm{i}, \quad (6)$$

we conclude that $|\xi\rangle$ and $|\eta\rangle$ are mutual conjugate states. The overlap between $\langle\eta|$ and $|\xi\rangle$ is [3]

$$\langle\eta|\,\xi\rangle = \frac{1}{2}\mathrm{e}^{\frac{\xi\eta^* - \xi^*\eta}{2}}. \quad (7)$$

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The transformation from $|\xi\rangle$ to $|\eta\rangle$ (or the transformation *vice versa*) can be considered a 2-dimensional Fourier transformation in complex form. $\langle\eta|\xi\rangle$ is the transformation kernel. Then a question naturally arises: Can this complex transformation be extended to the corresponding complex fractional Fourier transformation (CFFT)? As is well-known that the 1-dimensional Fractional Fourier Transformation (FFT) of real functions, which was firstly introduced mathematically in 1980 by Namias [4] and later by McBride and Kerr [5], has their optical implementation. In 1993, Mendlovic, Ozaktas and Lohmann [6,7] defined αth FFT physically, based on light propagation in quadratic graded-index media (GRIN media with medium parameters $n(r) = n_1 - n_2 r^2/2$), as follows: let the original function be input from one side of quadratic GRIN medium, at $z = 0$. Then, the light distribution observed at the plane $z = z_0$ corresponds to the α equal to the (z_0/L)th fractional Fourier transform of the input fraction, where $L \equiv (\pi/2)(n_1/n_2)^{1/2}$ is a characteristic distance. The α equal to the first Fourier transform, observed at $z_0 = L$, corresponds to the ordinary Fourier transform, by design. Another approach for introducing FFT was made by Lohmann [8] who pointed out the algorithmic isomorphism among image rotation, rotation of the Wigner distribution function, and fractional Fourier transforming. Lohmann proposed the FFT as the transform performed on a function that leads to a rotation with an angle of the associated Wigner distribution function.

Regarding the quantum homodyne tomography with use of the Wigner transformation we referee to the works of Smithey *et al.* [9] and Vogel and Risken [10].

In this work we shall study the complex fractional Fourier transform in the context of quantum mechanics, to be more concrete, we shall employ the EPR entangled state representation to establish a formulation of CFFT, which means that we shall link the fractional Fourier transformation of complex functions to appropriate operator transformation between two kinds of entangled state representations. In so doing, the CFFT of quantum mechanical wave functions can be derived more directly and concisely. Our work is arranged as follows: in Section 2 we briefly review the major properties of $|\eta\rangle$ and $|\xi\rangle$. In Section 3, starting from the entangled states we shall derive the quantum version of the transformation kernel of CFFT and define CFFT. In Section 4 we find the eigenmodes of CFFT which turns out to be two-variable Hermite polynomials. In Section 5 we derive CFFT of some quantum mechanical wave functions. Note that Fourier transforms are so important in modern optical engineering because they arise naturally in optical lens setup, we hope that CFFT introduced in this work may have widespread use in quantum optical information processing.

2 Basic properties of the entangled states $|\eta\rangle$ and $|\xi\rangle$

By using $|00\rangle\langle 00| =: \mathrm{e}^{-a_1^\dagger a_1 - a_2^\dagger a_2}:$, where $: \, :$ stands for the normal ordering, and the technique of integration within an ordered product (the technique of IWOP) of operators [11,12] we can prove the completeness relation of $|\eta\rangle$,

$$\int \frac{\mathrm{d}^2\eta}{\pi} |\eta\rangle\langle\eta| = \int \frac{\mathrm{d}^2\eta}{\pi} :\mathrm{e}^{-|\eta|^2+\eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger + \eta^* a_1 - \eta a_2 + a_1 a_2 - a_1^\dagger a_1 - a_2^\dagger a_2}:=1,$$

$$\mathrm{d}^2\eta = \mathrm{d}\eta_1 \mathrm{d}\eta_2, \quad (8)$$

where a_i and $a_i^\dagger$ are permutable within normal ordering symbol $: \, :$, so they can be viewed as C-numbers while the integration is going on, and orthonormal property

$$\langle\eta'|\,\eta\rangle = \pi\delta\left(\eta-\eta'\right)\delta\left(\eta^*-\eta'^*\right). \quad (9)$$

In a recent paper [13] the coordinate-momentum entanglement concept has been extended to number-difference-correlative amplitude based on the eigenvector equations [14]

$$\sqrt{\frac{a_1 - a_2^\dagger}{a_1^\dagger - a_2}}\,|\eta\rangle = \mathrm{e}^{\mathrm{i}\varphi}|\eta\rangle, \ \left(a_1 - a_2^\dagger\right)\left(a_1^\dagger - a_2\right)|\eta\rangle = |\eta|^2|\eta\rangle. \quad (10)$$

In a similar manner, we can prove that $|\xi\rangle$ is also complete and orthonormal

$$\int \frac{\mathrm{d}^2\xi}{\pi}|\xi\rangle\langle\xi| = 1, \ \langle\xi'|\,\xi\rangle = \pi\delta\left(\xi-\xi'\right)\delta\left(\xi^*-\xi'^*\right). \quad (11)$$

The overlap between the two-mode coherent state [15]

$$|\gamma,\beta\rangle = \exp\left[-\frac{|\gamma|^2+|\beta|^2}{2} + \gamma a_1^\dagger + \beta a_2^\dagger\right]|00\rangle \quad (12)$$

and the entangled states is

$$\langle\gamma',\beta'\,|\xi\rangle = \exp\left[-\frac{|\xi|^2}{2} + \xi\gamma'^* + \xi^*\beta'^* - \gamma'^*\beta'^* - \frac{|\gamma'|^2}{2} - \frac{|\beta'|^2}{2}\right],$$

$$\langle\eta|\,\gamma,\beta\rangle = \exp\left[-\frac{1}{2}|\eta|^2 + \eta^*\gamma - \eta\beta + \gamma\beta - \frac{|\gamma|^2}{2} - \frac{|\beta|^2}{2}\right]. \quad (13)$$

For later's use we list the Schmidt decomposition of the entangled states [16],

$$|\eta\rangle = \mathrm{e}^{-\mathrm{i}\eta_1\eta_2}\int_{-\infty}^{\infty}\mathrm{d}p\left|p+\sqrt{2}\eta_2\right\rangle_1 \otimes |-p\rangle_2\,\mathrm{e}^{-\mathrm{i}\sqrt{2}\eta_1 p},$$

$$|\xi\rangle = \mathrm{e}^{-\mathrm{i}\xi_1\xi_2}\int_{-\infty}^{\infty}\mathrm{d}x\,|x\rangle_1 \otimes \left|-x+\sqrt{2}\xi_1\right\rangle_2 \mathrm{e}^{\mathrm{i}\sqrt{2}x\xi_2}, \quad (14)$$

where $|x\rangle_i$ ($|p\rangle_i$) are coordinate (momentum) eigenstates.

3 Complex fractional Fourier transformation and EPR entangled states

Now we calculate the matrix element of $\exp[f(a_1^\dagger a_1 + a_2^\dagger a_2)]$ between $\langle\eta|$ and $|\xi\rangle$. Using (13) and the over-completeness relation of $|\gamma,\beta\rangle$,

$$\int \frac{d^2\gamma}{\pi}\frac{d^2\beta}{\pi}|\gamma,\beta\rangle\langle\gamma,\beta| = 1,$$

as well as the operator identity

$$e^{f a_1^\dagger a_1} = \sum_{n=0}^{\infty} e^{fn}|n\rangle_{11}\langle n| = \sum_{n=0}^{\infty} e^{fn}\frac{a_1^{\dagger n}}{\sqrt{n!}}|0\rangle_{11}\langle 0|\frac{a_1^n}{\sqrt{n!}} = \sum_{n=0}^{\infty} e^{fn} : \frac{\left(a_1^\dagger a_1\right)^n}{n!}e^{-a_1^\dagger a_1} :=: \exp\left[\left(e^f-1\right)a_1^\dagger a_1\right] :, \tag{15}$$

where $|n\rangle_1$ is the number state, we have

$$\begin{aligned}
&\langle\eta|\exp\left[f\left(a_1^\dagger a_1 + a_2^\dagger a_2\right)\right]|\xi\rangle = \\
&\langle\eta| : \exp\left[\left(e^f-1\right)\left(a_1^\dagger a_1 + a_2^\dagger a_2\right)\right] : |\xi\rangle \\
&= \int\frac{d^2\gamma}{\pi}\frac{d^2\beta}{\pi}\int\frac{d^2\gamma'}{\pi}\frac{d^2\beta'}{\pi}\langle\eta|\gamma,\beta\rangle\langle\gamma,\beta| \\
&\times : \exp\left[\left(e^f-1\right)\left(a_1^\dagger a_1 + a_2^\dagger a_2\right)\right] : |\gamma',\beta'\rangle\langle\gamma',\beta'|\xi\rangle \\
&= \int\frac{d^2\gamma}{\pi}\frac{d^2\beta}{\pi}\int\frac{d^2\gamma'}{\pi}\frac{d^2\beta'}{\pi}\exp\Bigg\{-|\eta|^2/2 + \eta^*\gamma - \eta\beta \\
&+ \gamma\beta - |\gamma|^2 - |\beta|^2 - |\xi|^2/2 + \xi\gamma'^* + \xi^*\beta'^* - \gamma'^*\beta'^* \\
&- |\gamma'|^2 - |\beta'|^2 + e^f\left(\gamma^*\gamma' + \beta^*\beta'\right)\Bigg\} = \frac{1}{1+e^{2f}} \\
&\times\exp\left(-\frac{|\eta|^2+|\xi|^2}{2} + \frac{e^f\left(|\eta|^2+|\xi|^2\right) + \left(\xi\eta^* - \eta\xi^*\right)}{e^{-f}+e^f}\right).
\end{aligned} \tag{16}$$

Let $f = i(\pi/2-\alpha)$, we have

$$\langle\eta|\exp\left[i(\pi/2-\alpha)\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right)\right]|\xi\rangle = \frac{1}{2\sin\alpha}\exp\left(\frac{i\left(|\eta|^2+|\xi|^2\right)}{2\tan\alpha} + \frac{\xi\eta^* - \eta\xi^*}{2\sin\alpha}\right). \tag{17}$$

Especially, when $\alpha = \pi/2$, equation (17) reduces to (7). Now we take the right hand side of (17) as the kernel of CFFT of α order, *i.e.*, we define αth CFFT of a complex function $g(\xi)$ as $F_\alpha(g(\xi))$ *via* the following relation,

$$F_\alpha(g(\xi)) = \frac{1}{2\sin\alpha}e^{i(\alpha-\pi/2)} \times\int\frac{d^2\xi}{\pi}\exp\left[\frac{i\left(|\eta|^2+|\xi|^2\right)}{2\tan\alpha} + \frac{\xi\eta^* - \xi^*\eta}{2\sin\alpha}\right]g(\xi) \equiv G(\eta). \tag{18}$$

We will show later that this CFFT can help us to reveal some new property which has been overlooked in the formulation of the direct product of two real FFTs. The definition (18) is of course required to satisfy the basic postulate that $F_\beta[F_\alpha g(\xi)] = F_{\beta+\alpha}(g(\xi))$ (the additivity property). For this purpose, we examine

$$\begin{aligned}
F_\beta F_\alpha(g(\xi)) &= F_\beta[G(\eta)] = \frac{1}{2\sin\beta}e^{i(\beta-\pi/2)} \\
&\times\int\frac{d^2\eta}{\pi}\exp\left[\frac{i\left(|\eta|^2+|\xi'|^2\right)}{2\tan\beta} + \frac{\eta\xi'^* - \eta^*\xi'}{2\sin\beta}\right]G(\eta) \\
&= \frac{1}{4\sin\alpha\sin\beta}e^{i(\alpha+\beta-\pi)} \\
&\times\int\frac{d^2\eta}{\pi}\exp\left[\frac{i\left(|\eta|^2+|\xi'|^2\right)}{2\tan\beta} + \frac{\eta\xi'^* - \eta^*\xi'}{2\sin\beta}\right] \\
&\times\int\frac{d^2\xi}{\pi}\exp\left[\frac{i\left(|\eta|^2+|\xi|^2\right)}{2\tan\alpha} + \frac{\xi\eta^* - \xi^*\eta}{2\sin\alpha}\right]g(\xi).
\end{aligned} \tag{19}$$

Using the formula

$$\cot\alpha - \sin\beta/[\sin\alpha\sin(\alpha+\beta)] = \cot\beta - \sin\alpha/[\sin\beta\sin(\alpha+\beta)] = \cot(\alpha+\beta), \tag{20}$$

Equation (19) becomes

$$\begin{aligned}
F_\beta F_\alpha(g(\xi)) &= \frac{1}{4\sin\alpha\sin\beta}\exp\left[\frac{i|\xi'|^2}{2\tan\beta}\right]e^{i(\alpha+\beta-\pi)} \\
&\times\int\frac{d^2\xi}{\pi}\exp\left[\frac{i|\xi|^2}{2\tan\alpha}\right]g(\xi)\int\frac{d^2\eta}{\pi} \\
&\times\exp\left[\frac{i|\eta|^2}{2}(\cot\beta+\cot\alpha) + \frac{\eta\xi'^* - \eta^*\xi'}{2\sin\beta} + \frac{\xi\eta^* - \xi^*\eta}{2\sin\alpha}\right] \\
&= \frac{1}{2\sin(\alpha+\beta)}e^{i(\alpha+\beta-\pi)}\exp\left[\frac{i|\xi'|^2}{2\tan(\alpha+\beta)}\right] \\
&\times\int\frac{d^2\xi}{\pi}\exp\left[\frac{i|\xi|^2}{2\tan(\alpha+\beta)} + \frac{i\left(\xi'\xi^* + \xi\xi'^*\right)}{2\sin(\alpha+\beta)}\right]g(\xi).
\end{aligned} \tag{21}$$

Let $\xi' \equiv i\eta'$, $\xi^{*\prime} = -i\eta'^*$, then $i(\xi'\xi^* + \xi\xi'^*) = \xi\eta'^* - \xi^*\eta'$, $|\xi'|^2 = |\eta'|^2$, in reference to the definition (18) we finally see

$$F_\beta F_\alpha(g(\xi)) = F_{\beta+\alpha}(g(\xi)) = G_{\beta+\alpha}(\eta'), \tag{22}$$

so the postulate is satisfied. As one can see shortly later, according to the definition (18) the CFFT of a two-variable Hermite polynomial is still a two-variable Hermite polynomial, so our definition of αth CFFT is also physically based on propagation in quadratic graded-index medium (GRIN), *i.e.*, the change of the light field caused by propagation along a quadratic GRIN medium by a length proportional to α.

4 Eigenmodes of CFFT

Since equation (17) is the quantum version of the transformation kernel of CFFT, it provides a new convenient way to calculate the CFFT of complex functions. That is, if we consider $g(\xi)$ as $\langle\xi|\, g\rangle$, the wave function of the state vector $|g\rangle$ in the $\langle\xi|$ representation, then from equation (17), and the completeness relation (11) we see that the CFFT (18) of $\langle\xi|\, g\rangle$ actually is

$$F_\alpha\ (g(\xi)) = \mathrm{e}^{\mathrm{i}(\alpha-\pi/2)} \times \int \frac{\mathrm{d}^2\xi}{\pi} \langle\eta| \exp\left[\mathrm{i}\,(\pi/2-\alpha)\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right)\right] |\xi\rangle \langle\xi|\, g\rangle$$
$$= \langle\eta| \exp\left[\mathrm{i}\,(\pi/2-\alpha)\left(a_1^\dagger a_1 + a_2^\dagger a_2\right)\right] |g\rangle \equiv G(\eta). \quad (23)$$

Moreover, let $G(\eta) \equiv \langle\eta|\, G\rangle$, equations (8, 23) imply

$$|G\rangle = \exp\left[\mathrm{i}\,(\pi/2-\alpha)\left(a_1^\dagger a_1 + a_2^\dagger a_2\right)\right] |g\rangle\,. \quad (24)$$

The formulas (11, 23) can help us to derive CFFT of some wave functions easily. For example, when $|g\rangle$ is a two-mode number state $|m,n\rangle = a_1^{\dagger m} a_2^{\dagger n}/\sqrt{m!n!}\,|0,0\rangle$, then the CFFT of the wave function $\langle\xi|\ m,n\rangle$ is

$$F_\alpha(\langle\xi|\ m,n\rangle) = \langle\eta| \exp\left[\mathrm{i}\,(\pi/2-\alpha)\left(a_1^\dagger a_1 + a_2^\dagger a_2\right)\right] |m,n\rangle$$
$$= \mathrm{i}^{n+m}\mathrm{e}^{-\mathrm{i}\alpha(m+n)} \langle\eta|\ m,n\rangle\,. \quad (25)$$

To calculate $\langle\eta|\ m,n\rangle$, let us recall the definition of two-variable Hermite polynomial $H_{m,n}(\zeta,\zeta^*)$ [17],

$$H_{m,n}(\lambda,\lambda^*) = \sum_{l=0}^{\min\ (m,n)} \frac{m!n!}{l!(m-l)!(n-l)!}(-1)^l \lambda^{m-l}\lambda^{*n-l}, \quad (26)$$

which is not a direct product of two independent single-variable Hermite polynomials. Using the generating function of $H_{m,n}(\lambda,\lambda^*)$,

$$\sum_{m,n=0}^{\infty} \frac{t^m t'^n}{m!n!} H_{m,n}(\lambda,\lambda^*) = \exp\{-tt' + t\lambda + t'\lambda^*\}, \quad (27)$$

we can expand $\langle\eta|$ as

$$\langle\eta| = \langle 00| \sum_{m,n=0}^{\infty} \mathrm{i}^{m+n} \frac{a_1^m a_2^n}{m!n!} H_{m,n}(-\mathrm{i}\eta^*, \mathrm{i}\eta)\mathrm{e}^{-|\eta|^2/2}, \quad (28)$$

thus

$$\langle\eta\ |m,n\rangle = \frac{\mathrm{i}^{m+n}}{\sqrt{m!n!}} H_{m,n}(-\mathrm{i}\eta^*, \mathrm{i}\eta)\mathrm{e}^{-|\eta|^2/2}. \quad (29)$$

On the other hand, from

$$\langle\xi| = \langle 00| \exp\left\{-a_1 a_2 + a_1\xi + a_2\xi^* - \frac{|\xi|^2}{2}\right\}$$
$$= \langle 00| \sum_{m,n=0}^{\infty} \frac{a_1^m a_2^n}{m!n!} H_{m,n}(\xi,\xi^*)\mathrm{e}^{-|\xi|^2/2}, \quad (30)$$

we have

$$\langle\xi\ |m,n\rangle = \frac{1}{\sqrt{m!n!}} H_{m,n}(\xi,\xi^*))\mathrm{e}^{-|\xi|^2/2}. \quad (31)$$

As a result of (29) and (31) we see that equation (25) becomes

$$F_\alpha\ (\mathrm{e}^{-|\xi|^2/2} H_{m,n}(\xi,\xi^*)) = (-\mathrm{e}^{-\mathrm{i}\alpha})^{m+n}\mathrm{e}^{-|\eta|^2/2} H_{m,n}(-\mathrm{i}\eta^*, \mathrm{i}\eta). \quad (32)$$

If we consider the operation F_α as an operator, one can say that the eigenfunction of F_α (the eigenmodes of CFFT) is the two-variable Hermite polynomials $H_{m,n}$ with the eigenvalue being $(-\mathrm{e}^{-\mathrm{i}\alpha})^{m+n}$. This is a new property of CFFT. Since the function space spanned by $H_{m,n}(\xi,\xi^*)$ is complete,

$$\int \frac{\mathrm{d}^2\xi}{\pi}\mathrm{e}^{-|\xi|^2} H_{m,n}(\xi,\xi^*) H^*_{m',n'}(\xi,\xi^*) = \sqrt{m!n!m'!n'!}\delta_{m',m}\delta_{n,n'}, \quad (33)$$

and

$$\sum_{m,n=0}^{\infty} \frac{1}{m!n!} H_{m,n}(\xi,\xi^*) \left[H_{m,n}(\xi',\xi'^*)\right]^* \mathrm{e}^{-|\zeta|^2} = \pi\delta\left(\xi-\xi'\right)\delta\left(\xi^*-\xi'^*\right), \quad (34)$$

one can confirms that the eigenmodes of CFFT form an orthogonal and complete basis set. Now let us recall one-dimensional real variable fractional Fourier transform (FFT) of α order, defined in [4–8] as

$$\mathtt{f}_\alpha(k(x)) = \sqrt{\frac{\exp\left[-\mathrm{i}\left(\pi/2-\alpha\right)\right]}{2\pi\sin\alpha}} \times \int_{-\infty}^{\infty} \exp\left\{\frac{\mathrm{i}(x^2+p^2)}{2\tan\alpha} - \frac{\mathrm{i}xp}{\sin\alpha}\right\} k\left(x\right)\mathrm{d}x \equiv \mathtt{G}(p) \quad (35)$$

(the usual Fourier transformation is a special case with order $\alpha = 1$). By direct checking one can see [6–8]

$$\mathtt{f}_\alpha(\mathrm{e}^{-x^2/2} H_n\left(x\right)) = \mathrm{e}^{-\mathrm{i}\alpha n}\mathrm{e}^{-p^2/2} H_n(p), \quad (36)$$

where H_n is the single-variable Hermite polynomials. Thus the eigenfunction of $\mathtt{f}_\alpha$(the eigenmodes of FFT) is the single-variable Hermite-Gauss function [6–8] with the eigenvalue being $\mathrm{e}^{-\mathrm{i}\alpha n}$. An extension to two lateral coordinates x and y is straightforwardly the direct product of two independent single-variable Hermite polynomials $H_n(x)\,H_m(y)$ as explained in [6]. Because the two-variable Hermite polynomials is not the direct product of two independent single-variable Hermite polynomials, it has been overlooked by the users of direct product of two single real-variable FFTs, only the CFFT defined as (18) helps to reveal the new property of CFFT.

Now let us compare the transformation kernel of CFFT with that of the direct product of two single real-variable FFTs in more detail. The latter in (35) can be proved equal to the following matrix elements in the momentum state ${}_i\langle p|$ and the coordinate state $|x\rangle_i$,

$$\prod_{i=1}^{2}\exp\left\{\frac{\mathrm{i}(x_i^2+p_i^2)}{2\tan\alpha}-\frac{\mathrm{i}x_ip_i}{\sin\alpha}\right\}=2\pi\mathrm{i}\sin\alpha\mathrm{e}^{-\mathrm{i}\alpha}{}_1\langle p|\otimes_2\langle p|$$
$$\times\exp\left[\mathrm{i}\left(\pi/2-\alpha\right)\left(a_1^\dagger a_1+a_2^\dagger a_2\right)\right]|x\rangle_1\otimes|x\rangle_2\,,\quad(37)$$

in contrast, from (14) we see that the transformation kernel of CFFT in (17) is equal to

$$\frac{1}{2\sin\alpha}\exp\left(\frac{\mathrm{i}\left(|\eta|^2+|\xi|^2\right)}{2\tan\alpha}+\frac{\xi\eta^*-\eta\xi^*}{2\sin\alpha}\right)$$
$$=\mathrm{e}^{\mathrm{i}\eta_1\eta_2-\mathrm{i}\xi_1\xi_2}\int_{-\infty}^{\infty}\mathrm{d}p_1\left\langle p+\sqrt{2}\eta_2\right|\otimes_2\langle -p|\,\mathrm{e}^{\mathrm{i}\sqrt{2}\eta_1 p}$$
$$\times\exp\left[\mathrm{i}\left(\pi/2-\alpha\right)\left(a_1^\dagger a_1+a_2^\dagger a_2+1\right)\right]$$
$$\times\int_{-\infty}^{\infty}\mathrm{d}x\,|x\rangle_1\otimes\left|-x+\sqrt{2}\xi_1\right\rangle_2\mathrm{e}^{\mathrm{i}\sqrt{2}x\xi_2}.\quad(38)$$

Therefore we see that, so far as the quantum version of transformation kernel is concerned, the CFFT is different from the direct product of two real FFTs

5 Some wave functions' CFFT

Equation (23) provides us with a new approach for calculating the CFFT of quantum mechanical wave functions in the $\langle\xi|$ representation. For instance, when $|g\rangle$ is a unnormalized two-mode coherent state $|\gamma,\beta\rangle=\exp[\gamma a_1^\dagger+\beta a_2^\dagger]|0,0\rangle$, then from (13) we see that the CFFT of this coherent state wave function is

$$F_\alpha\left(\langle\xi\,|\gamma,\beta\rangle\right)=\langle\eta|\exp\left[\mathrm{i}\left(\pi/2-\alpha\right)\left(a_1^\dagger a_1+a_2^\dagger a_2\right)\right]|\gamma,\beta\rangle$$
$$=\langle\eta\,\left|\mathrm{i}\mathrm{e}^{-\mathrm{i}\alpha}\gamma,\mathrm{i}\mathrm{e}^{-\mathrm{i}\alpha}\beta\right\rangle$$
$$=\exp\left[-\frac{|\eta|^2}{2}+\mathrm{i}\mathrm{e}^{-\mathrm{i}\alpha}\eta^*\gamma-\mathrm{i}\mathrm{e}^{-\mathrm{i}\alpha}\eta\beta\mathrm{e}^{-2\mathrm{i}\alpha}-\gamma\beta-\frac{|\gamma|^2}{2}-\frac{|\beta|^2}{2}\right].\quad(39)$$

When $|g\rangle$ is a two-mode squeezed vacuum state,

$$|g\rangle\rightarrow S\,|0,0\rangle=\sec h\lambda\exp\left[a_1^\dagger a_2^\dagger\tanh\lambda\right]|0,0\rangle\,,\quad(40)$$

where S is the squeezing operator, in $\langle\xi|$ representation it can be expressed as [18]

$$S=\sqrt{\mu}\int\frac{\mathrm{d}^2\xi}{\pi}\,|\xi\mu\rangle\,\langle\xi|\,,\quad\mu=\mathrm{e}^\lambda>0\quad(41)$$

then using $S|0,0\rangle=\sqrt{\mu}\int(\mathrm{d}^2\xi/\pi)|\xi\mu\rangle\mathrm{e}^{-|\xi|^2/2}$ and equation (23), the CFFT of the wave function of the squeezed state is

$$F_\alpha\left(\langle\xi|\,S\,|0,0\rangle\right)=\langle\eta|\exp\Big[\mathrm{i}\left(\pi/2-\alpha\right)$$
$$\times\left(a_1^\dagger a_1+a_2^\dagger a_2\right)\Big]S\,|0,0\rangle=\sqrt{\mu}\int\frac{\mathrm{d}^2\xi}{\pi}$$
$$\times\langle\eta|\exp[\mathrm{i}(\pi/2-\alpha)(a_1^\dagger a_1+a_2^\dagger a_2)]|\xi\mu\rangle\mathrm{e}^{-|\xi|^2/2}=\frac{-\mathrm{i}\sqrt{\mu}}{2\sin\alpha}\mathrm{e}^{\mathrm{i}\alpha}$$
$$\times\int\frac{\mathrm{d}^2\xi}{\pi}\exp\left(\frac{\mathrm{i}\left(|\eta|^2+|\mu\xi|^2\right)}{2\tan\alpha}+\mu\frac{\xi\eta^*-\eta\xi^*}{2\sin\alpha}\right)\mathrm{e}^{-|\xi|^2/2}$$
$$=\sqrt{\frac{\mu\mathrm{e}^{\mathrm{i}\alpha}}{\mathrm{i}\sin\alpha+\mu^2\cos\alpha}}\exp\left[\frac{\mathrm{i}-\mu^2\tan\alpha}{2\left(\tan\alpha-\mathrm{i}\mu^2\right)}|\eta|^2\right].\quad(42)$$

In summary, we have introduced the CFFT by virtue of the quantum mechanical operator transform in $|\xi\rangle-|\eta\rangle$ entangled representations. In this way the optical fractional Fourier transform of complex functions can be converted to its quantum mechanical correspondence. The link between the two aspects is established. The properties and the essence of CFFT can be seen more clearly from the point of view of entangled states representation transform in quantum mechanics. The CFFT of wave functions in the entangled state representation can be derived more directly and concisely. The whole discussion in this work explains the new application of entangled state representations in optical fractional Fourier transformation. We expect some optical lens-setup could be constructed by experimentalists to produce two-variable Hermite polynomial eigenmodes and to demonstrate some other CFFT functions in the future. Generally speaking, 2-dimensional FFT (CFFT in this paper is a 2-dimensional FFT in the convenient complex form) can be provided in a setup that involves free-space propagation-lens-free-space propagation or lens-free-space propagation–lens. In the practical realization by a lens, the conventional Fourier transform is scaled by the lens focal length, and therefore CFFT are also tied to it [19]. We also expect that the CFFT can play the role in transmitting quantum information, in constructing quantum imaging systems, once the CFFT of quantum mechanical wave functions can be experimentally realized.

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